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WEAKLY LOCALLY ARTINIAN SUPPLEMENTED MODULES

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Abstract. In this study, by using the concept of locally artinian supplemented modules, we have obtained weakly locally artinian supplemented modules as a proper generalization of these modules in module theory. Our results generalize and extend various comparable results in the existing literature. We have proved that the notion of weakly locally artinian supplemented modules inherited by factor modules, finite sums and small covers. We have obtained that weakly locally artinian supplemented modules with small radical coincide with weakly (radical) supplemented modules which have locally artinian radical. Also, we have shown that if N and $\frac{M}{N}$ are weakly locally artinian supplemented for some submodule $N \subseteq M$ which has a weak locally artinian supplement in M then M is weakly locally artinian supplemented.

1. Introduction

Throughout this paper, the ring R will denote an associative ring with identity element and modules will be left unital. We will use the notation $U \ll M$ to stress that U is a *small* submodule of M. A submodule $N \subseteq M$ is said to be *essential* in M, denoted as $N \trianglelefteq M$, if $N \cap K \neq 0$ for every non-zero submodule $K \subseteq M$. By $Rad(M)$ we denote the sum of all small submodules of M or, equivalently the intersection of all maximal submodules of M. $Soc(M)$ will indicate socle of M which is sum of all semisimple submodules of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M, and M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. A module M is called *semilocal* if $\frac{M}{Rad(M)}$ is semisimple. A ring R is said to be *semilocal* if $\frac{R}{Rad(R)}$ is semisimple. By [5, Proposition 20.2], a commutative ring R is semilocal if and only if R has only finitely many maximal ideals. M is called *locally artinian* if every finitely generated submodule of M is artinian [10, 31]. A submodule V of M is called *a supplement* of U in M if $M = U + V$ and $U \cap V \ll V$. A submodule V of M is called *a weak supplement* of U in M if $M = U + V$ and $U \cap V \ll M$. The module M is called *(weakly) supplemented* if every submodule of M has a (weak) supplement in M. In [1], it is proved that the class of weakly supplemented modules need not be

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closed under extensions, that is if U and M/U are weakly supplemented for some submodule U of M then M need not be weakly supplemented. A submodule U of M has ample supplements in M if every submodule V of M such that $M = U + V$ contains a supplement $V^{'}$ of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M [10].

Let R be a a principal ideal domain (PID) with exactly one non-zero maximal ideal, then R is said to be *a Discrete valuation ring (DVR)*. By [13, Lemma 2.1] that every module with small radical over a Discrete Valuation Ring, is the direct sum of a finitely generated free module and a bounded module. In [12], he generalized the concept of modules with small radical to radical supplemented modules. M is called *radical supplemented* if $Rad(M)$ has a supplement in M. These modules are also a proper generalization of supplemented modules. Then, in [2], it is defined as a module M *strongly radical supplemented* (or briefly srs) if every submodule N of M with $Rad(M) \subseteq N$ has a supplement in M. Then it is introduced that modules whose every submodule containing the radical has a weak supplement (in particular, over dedekind domains the radical has a weak supplement) in the module as *weakly radical supplemented module (wrs)* which is a generalization of strongly radical supplemented modules [7].

In [11], a generalization of concept of socle as a $Soc_s(M) = \sum\{ U \ll M \,|\, U$ is simple$\}$. Here $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$. In [3], a module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. A submodule U of M is called an *ss-supplement* of U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. The module M is called *ss-supplemented* if every submodule of M has an *ss*-supplement in M. A submodule U of M has ample *ss*-supplements in M if every submodule V of M such that $M = U + V$ contains an *ss*-supplement $V^{'}$ of U in M. The module M is called *amply ss-supplemented* if every submodule of M has ample *ss*-supplements in M. In [8], strongly local and (amply) *ss*-supplemented modules are generalized as RLA-local and (amply) locally artinian supplemented modules, respectively. A local module M is called *RLA-local* if $Rad(M)$ is a locally artinian submodule of M. A module M is called *locally artinian supplemented* if every submodule U of M has a locally artinian supplement in M, that is, V is a supplement of U in M such that $U \cap V$ is locally artinian. M is called *amply locally artinian supplemented* if every submodule U of M has ample locally artinian supplements in M. Here a submodule U of M has ample locally artinian supplements in M if every submodule V of M such that $M = U + V$ contains a locally artinian supplement $V^{'}$ of U in M.

In Section 2, it is proved that a module with a small radical is weakly locally artinian supplemented if and only if M is weakly supplemented and $Rad(M)$ is locally artinian. It is also proved that finite sum of weakly locally artinian supplemented modules is weakly locally artinian supplemented and every factor module of a weakly locally artinian supplemented module is weakly locally artinian supplemented. It is shown that a notion of weakly locally artinian supplemented modules inherited by small cover. It is also shown that if N and $\frac{M}{N}$ are weakly locally artinian supplemented for some submodule $N \subseteq M$ which has a weak locally artinian supplement in M, then M is weakly locally artinian supplemented.

2. Weakly Locally Artinian Supplemented Modules

Definition 1. Let M be a module. Then M is called *weakly locally artinian supplemented* if every submodule N of M has a weak supplement K in M with $N \cap K$ is locally artinian, i.e. N has a weak locally artinian supplement K in M.

By the definition, it is clear that every weakly locally artinian supplemented module is weakly supplemented. The following example shows that the converse is not always true.

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. By [1, Lemma 2.8], $M =_{\mathbb{Z}} \mathbb{Q}$ is weakly supplemented. So $\frac{\mathbb{Q}}{\mathbb{Z}}$ is weakly supplemented because of $\mathbb{Q}$ is weakly supplemented. But $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not locally artinian by [9, Theorem 3]. Since $Rad(\frac{\mathbb{Q}}{\mathbb{Z}}) = \frac{\mathbb{Q}}{\mathbb{Z}}$, $\frac{\mathbb{Q}}{\mathbb{Z}}$ is not weakly locally artinian supplemented.

Lemma 1. *Let M be a weakly supplemented module and $Rad(M)$ be a locally artinian submodule of M. Then M is weakly locally artinian supplemented.*

Proof. Let $N \subseteq M$. By the hypothesis, there exists a submodule K of M such that $M = N + K$, $N \cap K \ll M$. So $N \cap K \subseteq Rad(M)$. Since $Rad(M)$ is a locally artinian submodule of M, $N \cap K$ is a locally artinian submodule of M by [10, 31.2 (ii)]. Thus M is weakly locally artinian supplemented. □

Theorem 1. *Let M be a module with small radical. Then the following statements are equivalent.*

1. *M is weakly locally artinian supplemented;*
2. *M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M;*
3. *M is weakly supplemented and $Rad(M)$ is locally artinian.*

Proof. (1) ⇒ (2) Since M is weakly locally artinian supplemented, M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M.

(2) ⇒ (3) Since $Rad(M) \ll M$, M is a weak locally artinian supplement of $Rad(M)$ in M. Thus we have $M = Rad(M) + M$, $Rad(M) = Rad(M) \cap M \ll M$ and $Rad(M)$ is locally artinian.

(3) ⇒ (1) Clear by Lemma 1. □

Since every finitely generated module has a small radical, we have:

Corollary 1. Let M be a finitely generated module. Then M is weakly locally artinian supplemented if and only if M is weakly supplemented with locally artinian radical.

Proposition 1. Let M be a weakly locally artinian supplemented module and $N, K \subseteq M$ be submodules with $M = N + K$. Then K contains a weak locally artinian supplement $K^{'}$ of N in M.

Proof. Let $N \cap K = U$. Since M is weakly locally artinian supplemented, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll M$ and $U \cap V$ is locally artinian. Then $K = K \cap (U + V) = U + (K \cap V)$ and $M = N + K = N + [U + (K \cap V)] = N + (K \cap V)$. It follows that $N \cap (K \cap V) = U \cap V \ll M$ and $U \cap V$ is locally artinian, say $K^{'} = K \cap V$. Then we obtain that $K^{'}$ is a weak locally artinian supplement of N in M, as required. □

Proposition 2. Let M be a weakly locally artinian supplemented module and N be a small submodule of M. Then N is locally artinian.

Proof. By the hypothesis, there exists a submodule K of M such that $M = N + K$, $N \cap K \ll M$ and $N \cap K$ is locally artinian. Since $N \ll M$, $M = K$. So $N \cap K = N \cap M = N$ is locally artinian. □

Corollary 2. Let M be a weakly locally artinian supplemented module and $Rad(M) \ll M$. Then $Rad(M)$ is locally artinian.

Proof. Clear by Proposition 2. □

With the help of the next theorem, we verify that under special conditions, notions of weakly locally artinian supplemented modules and weakly radical supplemented modules are the same.

Theorem 2. *Let M be a module with $Rad(M) \ll M$. Then the following statements are equivalent.*

(1) *M is weakly locally artinian supplemented;*
(2) *M is weakly supplemented and $Rad(M)$ has a weak locally artinian supplement in M;*
(3) *M is weakly supplemented and $Rad(M)$ is locally artinian;*
(4) *M is weakly radical supplemented and $Rad(M)$ is locally artinian.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Clear by Theorem 1.

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Let $N \subseteq M$. By the hypothesis, $N + Rad(M)$ has a weak supplement K in M. Then we have $M = (N + Rad(M)) + K$. Since $Rad(M) \ll M$, $M = N + K$, $N \cap K \subseteq (N + Rad(M)) \cap K \ll M$. So $N \cap K \ll M$. Thus $N \cap K \subseteq Rad(M)$. Since $Rad(M)$ is locally artinian, $N \cap K$ is locally artinian by [10, 31.2 (ii)]. Therefore K is a weak locally artinian supplement of N in M, as desired. □

We will show that in the factor modules, the property is preserved in weakly locally artinian supplemented modules just as it is in weakly supplemented modules.

Proposition 3. If M is a weakly locally artinian supplemented module, then every factor module of M is weakly locally artinian supplemented.

Proof. Let M be a weakly locally artinian supplemented module and $\frac{M}{N}$ be a factor module of M. By the assumption, for any submodule $N \subseteq U \subseteq M$, there exists a submodule V of M such that $M = U + V$, $U \cap V \ll M$ and $U \cap V$ is locally artinian. Let $\pi : M \longrightarrow \frac{M}{N}$ be the canonical projection. Then we have $\frac{M}{N} = \frac{U}{N} + \frac{V+N}{N}$, $\frac{U}{N} \cap \frac{V+N}{N} = \pi(U \cap V) \ll \frac{M}{N}$ and $\frac{U}{N} \cap \frac{V+N}{N} = \pi(U \cap V)$ is locally artinian by [10, 31.2 (i)], as required. □

The following lemma plays a key role in showing that the notion of weakly locally artinian supplemented modules is inherited by finite sum.

Lemma 2. *Let M be a module, $M_1 \subseteq M$, $N \subseteq M$ and M_1 be weakly locally artinian supplemented. If $M_1 + N$ has a weak locally artinian supplement in M, then N has a weak locally artinian supplement in M.*

Proof. Let K be a weak locally artinian supplement of $M_1 + N$ in M. Then we can write $M = (M_1 + N) + K$, $(M_1 + N) \cap K \ll M$ and $(M_1 + N) \cap K$ is locally artinian.

Since M_1 is weakly locally artinian supplemented, $(N+K)\cap M_1$ has a weak locally artinian supplement L in M_1, i.e. $M_1 = (N+K)\cap M_1 + L$, $(N+K)\cap L \ll M_1$ and $(N+K)\cap L$ is locally artinian. Then $M = M_1 + (N+K) = [(N+K)\cap M_1 + L] + (N+K) = N + (K+L)$ and $N\cap(K+L) \subseteq K\cap(N+L) + L\cap(N+K) \subseteq K\cap(N+M_1) + L\cap(N+K) \ll M$. By [10, 31 (2) (i), (ii)], $N\cap(K+L)$ is locally artinian and so $K+L$ is a weak locally artinian supplement of N in M. $\square$

Corollary 3. Let M be an R-module, $N \subseteq M$ and $M_i \subseteq M$ for $i = 1, 2, \dots, n$. If $N + M_1 + \cdots + M_n$ has a weak locally artinian supplement in M and M_i is a weakly locally artinian supplemented module for every $i = 1, 2, \dots, n$, then N has a weak locally artinian supplement in M.

Corollary 4. Let $M = M_1 + M_2$. If M_1 and M_2 are weakly locally artinian supplemented modules, then M is a weakly locally artinian supplemented module.

The following corollary is obtained from the previous result by applying induction.

Corollary 5. A finite sum of weakly locally artinian supplemented modules is weakly locally artinian supplemented.

Recall from [6] that N is a *small cover* of a module M if there exists an epimorphism $f : N \longrightarrow M$ such that $Ker(f) \ll M$.

Lemma 3. *Let M be a weakly locally artinian supplemented module. Then every small cover of M is weakly locally artinian supplemented.*

Proof. Let N be a small cover of M. Then there exists an epimorphism $f : N \longrightarrow M$ such that $Ker(f) \ll N$. Note that $f^{-1}(K) \ll N$ for every $K \ll M$ holds since $Ker(f) \ll N$. Let $L \subset N$. Then $f(L)$ has a weak locally artinian supplement of X in M. Note that $M = X + f(L)$, $X \cap f(L) \ll M$ and $X \cap f(L)$ is locally artinian. Again it is easy to check that $f^{-1}(X)$ is a weak locally artinian supplement of L in N. $\square$

Proposition 4. Let M be a weakly locally artinian supplemented module. Then every locally artinian supplement in M is weakly locally artinian supplemented.

Proof. Let K be a locally artinian supplement of N in M. Then we have $M = N + K$, $N \cap K \ll K$ and $N \cap K$ is locally artinian. $\frac{M}{N} \cong \frac{K}{(N\cap K)}$ is weakly locally artinian supplemented by Proposition 3. By Lemma 3, K is weakly locally artinian supplemented. $\square$

Corollary 6. Let M be a weakly locally artinian supplemented module. Then every locally artinian direct summand in M is weakly locally artinian supplemented.

Proof. Since every locally artinian direct summand is locally artinian supplement, the proof follows from Proposition 4. $\square$

Recall that a submodule $N \subseteq M$ is called *closed* in M if $N \trianglelefteq K$ for some $K \subseteq M$ implies $K = N$. A submodule $N \subseteq M$ is called *coclosed* in M if $\frac{N}{K} \ll \frac{M}{K}$ for some $K \subseteq M$ implies $K = N$.

Theorem 3. *Let $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence. If K and N are weakly locally artinian supplemented and K has a weak locally artinian supplement in M then M is weakly locally artinian supplemented. If K*

is coclosed locally artinian submodule in M then the converse holds, that is if M is weakly locally artinian supplemented then K and N are weakly locally artinian supplemented.

Proof. Without restriction of generality we will assume that $K \subseteq M$. Let T be a weak locally artinian supplement of K in M i.e. $M = K+T$, $K\cap T \ll M$ and $K\cap T$ is locally artinian. Then we have, $\frac{M}{(K\cap T)} = \frac{K}{(K\cap T)} \oplus \frac{T}{(K\cap T)}$. Since $\frac{K}{(K\cap T)}$ is a factor module of K, $\frac{K}{(K\cap T)}$ is weakly locally artinian supplemented by Proposition 3. On the other hand, $\frac{T}{(K\cap T)} \cong \frac{M}{K} \cong N$ is weakly locally artinian supplemented by the hypothesis. Then, by Corollary 5, $\frac{M}{(K\cap T)}$ is weakly locally artinian supplemented as a finite sum of weakly locally artinian supplemented modules. It follows from Lemma 3 that M is weakly locally artinian supplemented.

Suppose that M is weakly locally-artinian supplemented and K is a coclosed locally-artinian submodule in M. Then $K \cap T \ll K$ by [4, Lemma 1.1] and $K \cap T$ is locally artinian by [10, 31.2 (ii)] i.e. K is a locally artinian supplement of T in M. Therefore K is weakly locally artinian supplemented by Proposition 4. □

Recall from [6, Theorem 3.5] that a ring R is semilocal if and only if every R-module with small radical is weakly supplemented. By using Theorem 1, we have the following Proposition.

Proposition 5. Let R be a semilocal ring and M be an R-module. Suppose $N \subseteq M$ such that $\frac{M}{N}$ is finitely generated and $Rad(\frac{M}{N})$ is locally artinian. If N is weakly locally artinian supplemented then M is weakly locally artinian supplemented.

Proof. Suppose $\frac{M}{N}$ is generated by $m_1+N, m_2+N, \ldots, m_n+N$. For the submodule $K = Rm_1 + Rm_2 + \cdots + Rm_n$, we have $M = N + K$. Then M is weakly locally artinian supplemented by Corollary 4. □

3. Conclusion

The aim of this paper is to reveal the existence of the concept of weakly locally artinian supplemented modules. Our results improve and generalize some known results on locally artinian supplemented modules.

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