Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.1141344
11 (3) 129-137 (2023) - Research Article
ISSN: 2147-6268

Interpolative KMK-Type Fixed-Figure Results

Nihal Taş*

Abstract
Fixed-figure problem has been introduced as a generalization of fixed circle problem and investigated a geometric generalization of fixed point theory. In this sense, we prove new fixed-figure results with some illustrative examples on metric spaces. For this purpose, we use KMK-type contractions, that is, Kannan type and Meir-Keeler type contractions.

Keywords: *Fixed figure; fixed point; KMK-type contraction; metric space.*

AMS Subject Classification (2020): *Primary: 54H25 ; Secondary: 47H09; 47H10.*

****Corresponding author***

1. Introduction

In recent years, fixed-point theory has been generalized using the geometric approaches. For this purpose, fixed-circle problem has been occurred as a geometric generalization to the fixed-point theory when the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ has more than one fixed point [1]. In many studies, there are different solutions to this problem with applications on metric and some generalized metric spaces (for example, see [2], [3], [4], [5], [6], [7], [8] and [9]). After than, this problem has been extended to fixed-figure problem [10]. For this problem, the following notions were defined (see [11], [12], [1] and [10]).

Let $(\mathfrak{X}, \mathfrak{d})$ be a metric space, $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ a self-mapping and $\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$, $\mathfrak{r} \in [0, \infty)$. Then,

(a) the circle $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}}$ is defined by

$$\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0) = \mathfrak{r}\}.$$

(b) the disc $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$ is defined by

$$\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0) \leq \mathfrak{r}\}.$$

(c) the ellipse $\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2) = \mathfrak{r}\}.$$

Received : *06-07-2022,* ***Accepted :*** *10-11-2022*

*(**Cite as** "N. Taş, Interpolative KMK-Type Fixed-Figure Results, Math. Sci. Appl. E-Notes, 11(3) (2023), 129-137")*

(d) the hyperbola $\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : |\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)| = \mathfrak{r}\}.$$

(e) the Cassini curve $\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\, \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2) = \mathfrak{r}\}.$$

(f) the Apollonius circle $\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ is defined by

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \left\{\mathfrak{x} \in \mathfrak{X} - \{\mathfrak{x}_2\} : \frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)} = \mathfrak{r}\right\}.$$

(g) the k-ellipse $\mathfrak{E}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}]$ is defined by

$$\mathfrak{E}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] = \left\{\mathfrak{x} \in \mathfrak{X} : \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i) = \mathfrak{r}\right\}.$$

A geometric figure $\mathcal{F}$ contained in the fixed point set $Fix(\mathfrak{T}) = \{\mathfrak{x} \in \mathfrak{X} : \mathfrak{x} = \mathfrak{T}\mathfrak{x}\}$ is called a *fixed figure* (a fixed circle, a fixed disc, a fixed ellipse, a fixed hyperbola, a fixed Cassini curve, etc.) of the self-mapping $\mathfrak{T}$ (see [10]). Some fixed-figure results were obtained using different aspects (see [13], [11], [12], [3], [10], [14] and [15] for more details).

In this paper, we investigate some solutions to the fixed-figure problem on metric spaces. To do this, we modify the Kannan type and Meir-Keeler type contractions used in the fixed-point theorems. We give some illustrative examples related to the proved fixed-figure results.

2. Main results

In this section, we present some solutions to the fixed-figure problem using Kannan type (see [16] and [17]) and Meir-Keeler type (see [18]) contractions on metric spaces. To do this, we inspire the used approaches in [19] and [20].

In the sequel, let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a self-mapping of a metric space $(\mathfrak{X}, \mathfrak{d})$ and the number $\mathfrak{r}$ defined as

$$\mathfrak{r} = \inf\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} \notin Fix(\mathfrak{T})\}. \tag{2.1}$$

Also, in the examples of this section, we use the usual metric $\mathfrak{d}$.

The following theorem can be considered as a new fixed-disc or fixed-circle theorem.

Theorem 2.1. *If there exist $\mathfrak{x}_0 \in \mathfrak{X}$ and $\gamma \in (0, 1)$ such that*

(a) There exists a $\delta(\mathfrak{r}) > 0$ so that

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_0) \leq \mathfrak{r},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$, then we have

(i) $\mathfrak{x}_0 \in Fix(\mathfrak{T})$,

(ii) $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$,

(iii) $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

Proof. (i) Let $\mathfrak{x}_0 \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we have

$$1 \leq \mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0) < [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0)]^{1-\gamma} = \mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x}_0),$$

a contradiction. So it should be $\mathfrak{x}_0 \in Fix(\mathfrak{T})$.

(ii) If $\mathfrak{r} = 0$, then we have $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} = \{\mathfrak{x}_0\}$ and from the condition (i), we get $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.2}$$

and by the condition (a), we have

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_0)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_0) \leq \mathfrak{r}. \tag{2.3}$$

If we combine the inequalities (2.2) and (2.3), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_0)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_0, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \leq \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$.

(iii) It can be easily seen that $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}} \subseteq Fix(\mathfrak{T})$ since $\mathfrak{C}_{\mathfrak{x}_0,\mathfrak{r}}$ is a boundary of $\mathfrak{D}_{\mathfrak{x}_0,\mathfrak{r}}$. □

Example 2.1. Let $\mathfrak{X} = \{-1, 0, 1, 2\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}x = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 1 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.1 for $\mathfrak{x}_0 = 0$, $\gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = 2\} = 1$$

and

$$Fix(\mathfrak{T}) = \{-1, 0, 1\}$$

Consequently, $0 \in Fix(\mathfrak{T})$, $\mathfrak{D}_{0,1} = \{-1, 0, 1\} \subseteq Fix(\mathfrak{T})$ and $\mathfrak{C}_{0,1} = \{-1, 1\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.2. *If there exist $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ and $\gamma \in (0, 1)$ such that*

(a) *There exists a $\delta(\mathfrak{r}) > 0$ so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} \quad &< \quad [\mathfrak{d}(\mathfrak{x}, T\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \quad \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) + \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) *$\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,*

then we have

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) + \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.4}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} \quad &< \quad [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \quad \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) + \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq r. \end{aligned} \tag{2.5}$$

If we combine the inequalities (2.4) and (2.5), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{E}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.2. Let $\mathfrak{X} = \{-1, 1, 2, 3\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & 1 & 2 & 3 \\ -1 & 1 & 2 & 1 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.2 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = 3\} = 2$$

and

$$Fix(\mathfrak{T}) = \{-1, 1, 2\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{E}_2(-1, 1) = \{-1, 1\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.3. *If there exist $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$, $\gamma \in (0, 1)$ and $\mathfrak{r} > 0$ such that*
(a) There exists a $\delta(\mathfrak{r}) > 0$ so that

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)\right|^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \left|\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)\right| \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)\right|^{\gamma} \left|\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) - \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})\right|^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,
then we have

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{x} \in \mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)\right|^{\gamma} \left|\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x}) - \mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})\right|^{1-\gamma} \tag{2.6}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left|\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)\right|^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \left|\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1) - \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)\right| \leq \mathfrak{r}. \end{aligned} \tag{2.7}$$

If we combine the inequalities (2.6) and (2.7), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{H}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.3. Let $\mathfrak{X} = \left\{-1, \frac{1}{2}1, 2, \frac{5}{2}, 3, 4\right\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & \frac{1}{2} & 1 & 2 & \frac{5}{2} & 3 & 4 \\ -1 & \frac{5}{2} & 1 & 2 & \frac{5}{2} & 3 & 4 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.3 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{1}{3}$ and $\delta(\mathfrak{r}) = 2$. Also, we have

$$\mathfrak{r} = \inf \left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{-1, 1, 2, \frac{5}{2}, 3, 4\right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{H}_2(-1, 1) = \left\{-1, 1, 2, \frac{5}{2}, 3, 4\right\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.4. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ *and* $\gamma \in (0,1)$ *such that*
(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,
(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,
then we have

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)]^{\gamma} [\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})]^{1-\gamma} \tag{2.8}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} [\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2) \leq \mathfrak{r}. \end{aligned} \tag{2.9}$$

If we combine the inequalities (2.8) and (2.9), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{C}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.4. Let $\mathfrak{X} = \left\{-\sqrt{3}, -1, 0, 1, \sqrt{3}, 2\right\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -\sqrt{3} & -1 & 0 & 1 & \sqrt{3} & 2 \\ -\sqrt{3} & 1 & 0 & 1 & \sqrt{3} & 0 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.4 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{8}{9}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf \left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{-\sqrt{3}, -1, 0, 1, \sqrt{3}\right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{C}_2(-1, 1) = \left\{-\sqrt{3}, \sqrt{3}\right\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.5. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathfrak{X}$ *and* $\gamma \in (0,1)$ *such that*
(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)}\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \frac{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)} \leq \mathfrak{r},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)}\right]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})}{\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})}\right]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2 \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) = \{\mathfrak{x}_1\} = \{\mathfrak{x}_2\}$. From the condition (c), we get

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2)$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_2)}\right]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}_1, \mathfrak{T}\mathfrak{x})}{\mathfrak{d}(\mathfrak{x}_2, \mathfrak{T}\mathfrak{x})}\right]^{1-\gamma} \tag{2.10}$$

and by the condition (a), we have

$$\frac{\mathfrak{r}}{2} < [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\frac{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{x}, \mathfrak{x}_2)}\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \Longrightarrow \frac{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_1)}{\mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_2)} \leq \mathfrak{r}. \tag{2.11}$$

If we combine the inequalities (2.10) and (2.11), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{A}_{\mathfrak{r}}(\mathfrak{x}_1, \mathfrak{x}_2) \subseteq Fix(\mathfrak{T}).$$

□

Example 2.5. Let $\mathfrak{X} = \left\{-1, 0, \frac{1}{3}, 1, 2, 3\right\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}x = \begin{pmatrix} -1 & 0 & \frac{1}{3} & 1 & 2 & 3 \\ -1 & 0 & \frac{1}{3} & 1 & 0 & 3 \end{pmatrix},$$

for all $x \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.5 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 1$, $\gamma = \frac{8}{9}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf\left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \left\{-1, 0, \frac{1}{3}, 1, 3\right\}$$

Consequently, $-1, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{A}_2(-1, 1) = \left\{\frac{1}{3}, 3\right\} \subseteq Fix(\mathfrak{T})$.

Theorem 2.6. *If there exist* $\mathfrak{x}_1, \mathfrak{x}_2, \ldots \mathfrak{x}_k \in \mathfrak{X}$ *and* $\gamma \in (0, 1)$ *such that*

(a) *There exists a* $\delta(\mathfrak{r}) > 0$ *so that*

$$\begin{aligned} \frac{\mathfrak{r}}{2} &< [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \\ &\Longrightarrow \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i) \leq \mathfrak{r}, \end{aligned}$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(b)

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_i)\right]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma},$$

for all $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$,

(c) $\mathfrak{x}_1, \mathfrak{x}_2, \ldots \mathfrak{x}_k \in Fix(\mathfrak{T})$,

then we have

$$\mathfrak{C}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] \subseteq Fix(T).$$

Proof. Let $\mathfrak{r} = 0$. Then we have $\mathfrak{C}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] = \{\mathfrak{x}_1\} = \ldots = \{\mathfrak{x}_k\}$. From the condition (c), we get

$$\mathfrak{C}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] \subseteq Fix(\mathfrak{T}).$$

Let $\mathfrak{r} > 0$ and $\mathfrak{x} \in \mathfrak{C}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}]$ such that $\mathfrak{x} \in \mathfrak{X} - Fix(\mathfrak{T})$. Using the condition (b), we get

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}_i)\right]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} \tag{2.12}$$

and by the condition (a), we have

$$\begin{aligned} \frac{\mathfrak{r}}{2} \quad &< \quad [\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x})]^{\gamma} \left[\sum_{i=1}^{k} \mathfrak{d}(\mathfrak{x}, \mathfrak{x}_i)\right]^{1-\gamma} < \frac{\mathfrak{r}}{2} + \delta(\mathfrak{r}) \qquad (2.13) \\ \Longrightarrow \quad & \sum_{i=1}^{k} \mathfrak{d}(\mathfrak{T}\mathfrak{x}, \mathfrak{x}_i) \leq \mathfrak{r}. \end{aligned}$$

If we combine the inequalities (2.12) and (2.13), we obtain

$$1 \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) < \mathfrak{r} \leq \mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}),$$

a contradiction. It should be $\mathfrak{x} \in Fix(\mathfrak{T})$. Consequently, we get

$$\mathfrak{C}[\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_k; \mathfrak{r}] \subseteq Fix(\mathfrak{T}).$$

□

Example 2.6. Let $\mathfrak{X} = \{-1, 0, 1, 2\}$. Define the self-mapping $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ as

$$\mathfrak{T}\mathfrak{x} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 \end{pmatrix},$$

for all $\mathfrak{x} \in \mathfrak{X}$. Then $\mathfrak{T}$ validates the hypotheses of Theorem 2.6 for $\mathfrak{x}_1 = -1$, $\mathfrak{x}_2 = 0$, $\mathfrak{x}_3 = 1$, $\gamma = \frac{1}{2}$ and $\delta(\mathfrak{r}) = 4$. Also, we have

$$\mathfrak{r} = \inf \left\{\mathfrak{d}(\mathfrak{x}, \mathfrak{T}\mathfrak{x}) : \mathfrak{x} = \frac{1}{2}\right\} = 2$$

and

$$Fix(\mathfrak{T}) = \{-1, 0, 1\}$$

Consequently, $-1, 0, 1 \in Fix(\mathfrak{T})$ and $\mathfrak{C}[-1, 0, 1; 2] = \{0\} \subseteq Fix(\mathfrak{T})$.

3. Conclusion and future works

This paper is an example of the geometric approaches to fixed-point theory. The aim of this paper is to gain new solutions to the fixed-figure problem. For this paper, we use KMK-type contractions, that is, Kannan type and Meir-Keeler type contractions on metric spaces. This problem can be studied with different approaches on both metric spaces and some generalized metric spaces (for example, see [21], [22], [23] and the references therein).

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Özgür, N. Y., Taş, N.: *Some fixed-circle theorems on metric spaces*. Bulletin of the Malaysian Mathematical Sciences Society. **42** (4), 1433-1449 (2019).

[2] Kaplan, E., Mlaiki, N., Taş, N., Haque, S., Souayah, A. K.: *Some fixed-circle results with different auxiliary functions*. Journal of Function Spaces. **2022**, 2775733 (2022).

[3] Mlaiki, N., Özgür, N., Taş, N.: *New fixed-circle results related to F_c-contractive and F_c-expanding mappings on metric spaces*. Preprint arxiv:2101.10770 (2021).

[4] Özgür, N. Y., Taş, N.: *Some fixed-circle theorems and discontinuity at fixed circle*. AIP Conference Proceedings. **1926**, 020048 (2018).

[5] Özgür, N.: *Fixed-disc results via simulation functions*. Turkish Journal of Mathematics. **43** (6), 2794-2805 (2019).

[6] Pant, R. P., Özgür, N. Y., Taş, N.: *Discontinuity at fixed points with applications*. Bulletin of the Belgian Mathematical Society - Simon Stevin. **26**, 571-589 (2019).

[7] Pant, R. P., Özgür, N. Y., Taş, N.: *On discontinuity problem at fixed point*. Bulletin of the Malaysian Mathematical Sciences Society. **43**, 499-517 (2020).

[8] Pant, R. P., Özgür, N., Taş, N., Pant, A., Joshi, M. C.: *New results on discontinuity at fixed point*. Journal of Fixed Point Theory and Applications. **22**, 39 (2020).

[9] Taş, N.: *Bilateral-type solutions to the fixed-circle problem with rectified linear units application*. Turkish Journal of Mathematics. **44** (4), 1330-1344 (2020).

[10] Özgür, N., Taş, N.: *Geometric properties of fixed points and simulation functions*. Preprint arxiv:2102.05417 (2021).

[11] Erçınar, G. Z.: Some geometric properties of fixed points. Ph.D. thesis. Eskişehir Osmangazi University (2020).

[12] Joshi, M., Tomar, A., Padaliya, S. K.: *Fixed point to fixed ellipse in metric spaces and discontinuous activation function*. Applied Mathematics E-Notes. **21**, 225-237 (2021).

[13] Aytimur, H., Taş, N.: *A geometric interpretation to fixed-point theory on S_b-metric spaces*. Electronic Journal of Mathematical Analysis and Applications. **10** (2), 95-104 (2022).

[14] Taş, N., Özgür, N.: New fixed-figure results on metric spaces. In: Fixed point theory and fractional calculus - Recent advances and applications. Springer, Singapore 33-62 (2022).

[15] Aytimur, H., Güvenç, Ş., Taş, N.: *New fixed figure results with the notion of k-ellipse*. Mathematica Moravica. **27** (1), 37-52 (2023).

[16] Kannan, R.: *Some results on fixed points*. Bulletin of the Calcutta Mathematical Society. **60**, 71-76 (1968).

[17] Kannan, R.: *Some results on fixed points*. II. The American Mathematical Monthly. **76**, 405-408 (1969).

[18] Meir, A., Keeler, E.: *A theorem on contraction mappings*. Journal of Mathematical Analysis and Applications. **28**, 326-329 (1969).

[19] Karapınar, E.: *Revisiting the Kannan type contractions via interpolation*. Advances in the Theory of Nonlinear Analysis and its Applications. **2** (2), 85-87 (2018).

[20] Karapınar, E.: *Interpolative Kannan- Meir-Keeler type contraction*. Advances in the Theory of Nonlinear Analysis and its Applications. **5** (4), 611-614 (2021).

[21] Ege, O.: *Complex valued rectangular b-metric spaces and an application to linear equations*. Journal of Nonlinear Sciences and Applications. **8** (6), 1014-1021 (2015).

[22] Ege, O.: *Complex valued G_b-metric spaces*. Journal of Computational Analysis and Applications. **21** (2), 363-368 (2016).

[23] Gupta, V., Ege, O., Saini, R., Sen, M. D. L.: *Various fixed point results in complete G_b-metric spaces*. Dynamic Systems and Applications. **30** (2), 277-293 (2021).

Affiliations

Nihal Taş
Address: Balıkesir University, Dept. of Mathematics, 10145, Balıkesir-Türkiye.
e-mail: nihaltas@balikesir.edu.tr
ORCID ID: 0000-0002-4535-4019