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Exponential Function-Based Similarity Measures for *q*-Rung Linear Diophantine Fuzzy Sets and Their Application to Clustering Problem

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Highlights

- This paper focuses on the theory of q-rung linear Diophantine fuzzy sets (q-RLDFSs).
- Similarity measure approaches are proposed for determining closeness between two q-RLDFSs.
- A clustering algorithm based on the developed similarity measures of q-RLDFSs is constructed.

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Abstract

The q-rung linear Diophantine fuzzy set is a recently developed tool to handle with uncertain and vague information in real-life issues and can be applied for reference parameter-based opinions. Similarity measures determine distance with dimensions that represent features of the objects. Despite the importance of exponential function-based similarity measures, there is no satisfactory formulation for q-rung linear Diophantine fuzzy sets in the literature. This paper proposes similarity measures based on exponential function for q-rung linear Diophantine fuzzy sets and thus presents the first formulas for calculating the similarity coefficient between two q-rung linear Diophantine fuzzy sets. The salient features of the new similarity measures are axiomatically addressed to ensure their good performance. Also, they are applied to the clustering problem and the results are analyzed. A comparative study is established and thus several advantages of the proposed similarity measures are discussed.

1. INTRODUCTION

The fuzzy set theory has received increasing attention since its introduction by Zadeh [1]. By integrating the non-membership degree in this set structure, various fuzzy extensions have been developed including intuitionistic fuzzy set (IFS) [2], Pythagorean fuzzy set (PyFS) [3] and q-rung orthopair fuzzy set (q-ROFS) [4]. These FS extensions have numerous implementations in various real-life areas. Petchimuthu et al. [5] derived mean operators and some generalised products for matrix forms of fuzzy sets. Kamacı [6] studied a new extended kind of IFSs and presented its real-life application. In 2019, Jamkhaneh [7] introduced the modified modal operators on generalized IFSs. Chauan et al. [8] focused on intuitionistic fuzzy metric spaces and presented fixed point theorems in this metric spaces. Peng et al. [9,10] detailed some theoretical findings and future directions for PyFSs. In [11,12], the authors enriched and refined the q-ROFSs in theory and practice. Akram et al. [13,14] and Liu et al. [15] interested in q-rung type extensions of FSs and proposed correlational decision making frameworks under these set structures. Some authors studied the different types of the similarity measures (SMs) for the IFSs, PyFSs and q-ROFSs [16-19]. Thus, they endeavored to measure the intuitionistic fuzzy information, the Pythagorean fuzzy information and the q-rung orthopair fuzzy information. However, the above-mentioned FS extensions have their own constraints regarding membership degree and non-membership degree. To remove these constraints, Riaz and Hashmi [20] presented the idea of linear Diophantine fuzzy set (LDFS) by adding reference parameters. They argued that this idea eradicates the constraints of the present methodologies and the experts can freely determine the degrees without any constraints. Kamacı [21,22] derived some algebraic structures and complex forms of LDFSs.

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In 2021, Almagrabi et al. [23] put forward that the sum of the reference parameters, where an alternative satisfies the attribute acquired by expert is wider than one, so LDFS fails to achieve his/her goal regarding reference parameters. Therefore, they proposed a novel generalization of LDFSs, named q-rung linear Diophantine fuzzy set (q-RLDFS) and covered its vital properties. A q-RLDFS is superior to an LDFS because it is a q-RLDFS, but not conversely. The SM is an essential tool for measuring the uncertain information and is very useful in fields, such as clustering problem, multi-criteria decision making, game theory, pattern recognition, machine learning, medical diagnosis and etc.

To the best of our knowledge, determining the closeness between two q-RLDFSs is a research gap in the literature that needs to be filled, and this contribution provides the evaluation of q-RLDF-based information in many fields. The motivation of this article is to propose similarity measure models that simultaneously account for the vague degrees and the reference parameters of q-RLDFS. Following the above motivation, in this paper, exponential similarity measures (ESMs) of q-RLDFSs are developed. Also, new clustering models are created using these proposed ESMs, and then clustering problems are addressed in a modernize way. In practice, the calculation results of current similarity measures are prone to errors, hence good practice benefits are difficult to derive. Therefore, this paper focuses on new similarity measures based on exponential function (EF) and confirms that they can effectively handle the above-mentioned problem with comparative example and simulation situations.

The rest of this paper is systematized as follows: In section 2, we review some fundamental concepts related to the LDFSs and q-RLDFSs. In section 3, we procure the EF-based SMs for q-RLDFSs and present some of their theoretical results. In section 4, a clustering algorithm (CA) based on the novel SMs of q-RLDFSs is elaborated and applied to the clustering problem. Section 5 is reserved for comparison analysis, followed by the conclusions in section 6.

2. PRELIMINARIES

In this part, we recall the concepts of LDFSs, q-RLDFSs and some fundamental q-RLDFS operations.

In 2019, Riaz and Hashmi [20] proposed LDFS which is an extended forms of the IFS [2], PyFS [3] and q-ROFS [4], by including the degrees of reference/control parameters to the degrees of membership and non-membership. The concept of LDFS is given as follows.

Definition 2.1. ([20]) An LDFS $\mathfrak{X}$ in the universe discourse set $\mathfrak{D}$ is defined as

$$\mathfrak{X} = \{(d_k, \langle\Psi_{\mathfrak{X}}(d_k), \Theta_{\mathfrak{X}}(d_k)\rangle, \langle\alpha, \beta\rangle): d_k \in \mathfrak{D}\} \tag{2.1}$$

where $\Psi_{\mathfrak{X}}(d_k)$, $\Theta_{\mathfrak{X}}(d_k)$, α, $\beta \in [0,1]$ mean the grades of membership, non-membership and reference parameters of $d_k \in \mathfrak{D}$ into the set $\mathfrak{X}$. These fulfill the restriction $0 \leq \alpha\Psi_{\mathfrak{X}}(d_k) + \beta\Theta_{\mathfrak{X}}(d_k) \leq 1\ \forall\ d_k \in \mathfrak{D}$with $0 \leq \alpha + \beta \leq 1$.

Riaz and Hashmi [20] asserted that the reference/control parameters can help to identify or describe a particular model.

Almagrabi et al. [23] argued that in some real-life problems, the sum of the reference parameters might be larger than one, so LDFS cannot serve such purpose regarding reference parameters. To eliminate this contradiction, they described q-RLDFSs as follows.

Definition 2.2. ([23]**)** A q-RLDFS $\breve{\mathfrak{X}}$ in the universe discourse set $\mathfrak{D}$ is defined as

$$\breve{\mathfrak{X}} = \{\left(d_k, \langle\Psi_{\breve{\mathfrak{X}}}(d_k), \Theta_{\breve{\mathfrak{X}}}(d_k)\rangle, \langle\alpha, \beta\rangle\right): d_k \in \mathfrak{D}\} \tag{2.2}$$

where $\Psi_{\breve{\mathfrak{X}}}(d_k)$, $\Theta_{\breve{\mathfrak{X}}}(d_k)$, α, $\beta \in [0,1]$ mean the grades of membership, non-membership and reference parameters of $d_k \in \mathfrak{D}$ into the set $\breve{\mathfrak{X}}$, respectively. These fulfill the restriction $0 \leq \alpha^q \Psi_{\breve{\mathfrak{X}}}(d_k) + \beta^q \Theta_{\breve{\mathfrak{X}}}(d_k) \leq 1 \; \forall d_k \in \mathfrak{D}$ with $0 \leq \alpha^q + \beta^q \leq 1$ where $q \geq 1$. For each element q-RLDFS $\breve{\mathfrak{X}}$, the component $(\langle \Psi_{\breve{\mathfrak{X}}}(d_k), \Theta_{\breve{\mathfrak{X}}}(d_k) \rangle, \langle \alpha, \beta \rangle)$ is called a q-rung linear Diophantine fuzzy number (q-RLDFN) and simply q-RLDFN can be represented as $\eta = (\langle \Psi_{\breve{\mathfrak{X}}}, \Theta_{\breve{\mathfrak{X}}} \rangle, \langle \alpha, \beta \rangle)$.The set of all q-RLDFNs on $\mathfrak{D}$ is denoted by $q - RLDFN(\mathfrak{D})$. It is obvious that q-RLDFS reduces to LDFS when $q = 1$. Figure 1 presents the relationship between q-RLDFS and some existing fuzzy sets.

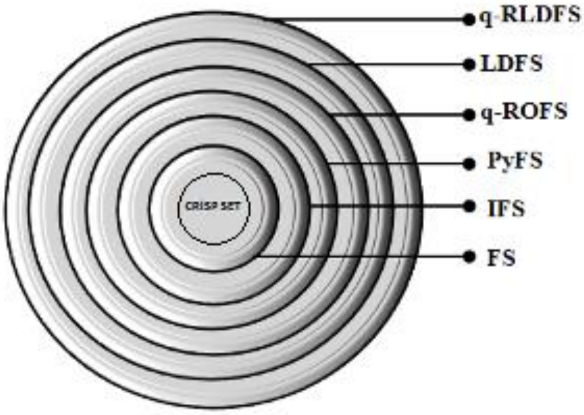

Figure 1. *Comparison view of q-RLDFS with some existing fuzzy sets*

Note that the reference/control parameters in the structure of q-RLDFS (or LDFS) are specified attributes, but their grades vary for each object in the universe discourse set $\mathfrak{D}$. The definition above may seem to imply that the grades of α and β are fixed for all objects, but they are not. To stress that the grades of reference parameters α and β for objects may vary and to better explain the notions in the next sections, the concept of q-LDFS is revisited as

$$\mathfrak{X} = \{(d_k, \langle \Psi_{\mathfrak{X}}(d_k), \Theta_{\mathfrak{X}}(d_k) \rangle, \langle \alpha_{\mathfrak{X}}(d_k), \beta_{\mathfrak{X}}(d_k) \rangle) : d_k \in \mathfrak{D}\} \tag{2.3}$$

Definition 2.3. ([23]**)** Let $\breve{\mathfrak{X}}_1 = \{(d_k, \langle \Psi_{\breve{\mathfrak{X}}_1}(d_k), \Theta_{\breve{\mathfrak{X}}_1}(d_k) \rangle, \langle \alpha_{\breve{\mathfrak{X}}_1}(d_k), \beta_{\breve{\mathfrak{X}}_1}(d_k) \rangle) : d_k \in \mathfrak{D}\}$ and $\breve{\mathfrak{X}}_2$ $\{(d_k, \langle \Psi_{\breve{\mathfrak{X}}_2}(d_k), \Theta_{\breve{\mathfrak{X}}_2}(d_k) \rangle, \langle \alpha_{\breve{\mathfrak{X}}_2}(d_k), \beta_{\breve{\mathfrak{X}}_2}(d_k) \rangle) : d_k \in \mathfrak{D}\}$ be two q-RLDFSs in the universal discourse set $\mathfrak{D}$. Then we have

a) $\breve{\mathfrak{X}}_1 \subseteq \breve{\mathfrak{X}}_2 \Leftrightarrow \Psi_{\breve{\mathfrak{X}}_1}(d_k) \leq \Psi_{\breve{\mathfrak{X}}_2}(d_k), \Theta_{\breve{\mathfrak{X}}_1}(d_k) \geq \Theta_{\breve{\mathfrak{X}}_2}(d_k), \alpha_{\breve{\mathfrak{X}}_1}(d_k) \leq \alpha_{\breve{\mathfrak{X}}_2}(d_k), \beta_{\breve{\mathfrak{X}}_1}(d_k) \geq \beta_{\breve{\mathfrak{X}}_2}(d_k)$ for all $d_k \in \mathfrak{D}$.

b) $\breve{\mathfrak{X}}_1 = \breve{\mathfrak{X}}_2 \Leftrightarrow \Psi_{\breve{\mathfrak{X}}_1}(d_k) = \Psi_{\breve{\mathfrak{X}}_2}(d_k), \Theta_{\breve{\mathfrak{X}}_1}(d_k) = \Theta_{\breve{\mathfrak{X}}_2}(d_k), \alpha_{\breve{\mathfrak{X}}_1}(d_k) = \alpha_{\breve{\mathfrak{X}}_2}(d_k), \beta_{\breve{\mathfrak{X}}_1}(d_k) = \beta_{\breve{\mathfrak{X}}_2}(d_k)$ for all $d_k \in \mathfrak{D}$.

Definition 2.4. ([23]**)** Let $\eta_1 = (\langle {}^1\Psi_{\breve{\mathfrak{X}}}, {}^1\Theta_{\breve{\mathfrak{X}}} \rangle, \langle {}^1\alpha_{\breve{\mathfrak{X}}}, {}^1\beta_{\breve{\mathfrak{X}}} \rangle)$and $\eta_2 = (\langle {}^2\Psi_{\breve{\mathfrak{X}}}, {}^2\Theta_{\breve{\mathfrak{X}}} \rangle, \langle {}^2\alpha_{\breve{\mathfrak{X}}}, {}^2\beta_{\breve{\mathfrak{X}}} \rangle)$ be two q-RLDFNs in the universal discourse set $\mathfrak{D}$ and the scalar $\zeta > 0$ then the following characteristics are valid;

a) $$\eta_1 \oplus \eta_2 = \begin{pmatrix} \langle \sqrt[q]{({}^1\Psi_{\breve{\mathfrak{X}}})^q + ({}^2\Psi_{\breve{\mathfrak{X}}})^q - ({}^1\Psi_{\breve{\mathfrak{X}}})^q ({}^2\Psi_{\breve{\mathfrak{X}}})^q}, \; {}^1\Theta_{\breve{\mathfrak{X}}} \, {}^2\Theta_{\breve{\mathfrak{X}}} \rangle, \\ \langle \sqrt[q]{({}^1\alpha_{\breve{\mathfrak{X}}})^q + ({}^2\alpha_{\breve{\mathfrak{X}}})^q - ({}^1\alpha_{\breve{\mathfrak{X}}})^q ({}^2\alpha_{\breve{\mathfrak{X}}})^q}, \; {}^1\beta_{\breve{\mathfrak{X}}} \, {}^2\beta_{\breve{\mathfrak{X}}} \rangle \end{pmatrix}; q \geq 1.$$

b) $$\eta_1 \otimes \eta_2 = \begin{pmatrix} \langle\ {}^1\Psi_{\breve{\mathfrak{X}}}\ {}^2\Psi_{\breve{\mathfrak{X}}}, \sqrt[q]{(\ {}^1\Theta_{\breve{\mathfrak{X}}})^q + (\ {}^2\Theta_{\breve{\mathfrak{X}}})^q - (\ {}^1\Theta_{\breve{\mathfrak{X}}})^q(\ {}^2\Theta_{\breve{\mathfrak{X}}})^q}\rangle \\ \langle\ {}^1\alpha_{\breve{\mathfrak{X}}}\ {}^2\alpha_{\breve{\mathfrak{X}}}, \sqrt[q]{(\ {}^1\beta_{\breve{\mathfrak{X}}})^q + (\ {}^2\beta_{\breve{\mathfrak{X}}})^q - (\ {}^1\beta_{\breve{\mathfrak{X}}})^q(\ {}^2\beta_{\breve{\mathfrak{X}}})^q}\rangle \end{pmatrix}; q \geq 1.$$

c) $$\zeta\eta_1 = \left(\langle\sqrt[q]{1-(1-(\ {}^1\Psi_{\breve{\mathfrak{X}}})^q)^\zeta}, (\ {}^1\Theta_{\breve{\mathfrak{X}}})^\zeta\rangle, \langle\sqrt[q]{1-(1-(\ {}^1\alpha_{\breve{\mathfrak{X}}})^q)^\zeta}, (\ {}^1\beta_{\breve{\mathfrak{X}}})^\zeta\rangle\right); q \geq 1.$$

d) $$\eta_1^\zeta = \left(\langle(\ {}^1\Psi_{\breve{\mathfrak{X}}})^\zeta, \sqrt[q]{1-(1-(\ {}^1\Theta_{\breve{\mathfrak{X}}})^q)^\zeta}\rangle, \langle(\ {}^1\alpha_{\breve{\mathfrak{X}}})^\zeta, \sqrt[q]{1-(1-(\ {}^1\beta_{\breve{\mathfrak{X}}})^q)^\zeta}\rangle\right); q \geq 1.$$

Definition 2.5. ([23]**)** Let $\eta_i = (\langle\ {}^i\Psi_{\breve{\mathfrak{X}}},\ {}^i\Theta_{\breve{\mathfrak{X}}}\rangle, \langle\ {}^i\alpha_{\breve{\mathfrak{X}}},\ {}^i\beta_{\breve{\mathfrak{X}}}\rangle)$ for $i \in I$ be an assembling of q-RLDFNs in the universal discourse set $\mathfrak{D}$ and the weight vector $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_m)^T$ where $\mathfrak{w}_i \in (0,1]$ for all $i \in I$ and $\sum_{i=1}^m \mathfrak{w}_i = 1$. The transformation $q - RLDFWAA_{\mathfrak{w}}: q - RLDFN(\mathfrak{D}) \to q - RLDFN(\mathfrak{D})$ is termed to be a q-rung linear Diophantine fuzzy weighted averaging aggregation (q-RLDFWAA) operator and described as

$$q - RLDFWAA_{\mathfrak{w}}(\eta_1, \eta_2, \dots, \eta_m) = \prod_{i=1}^m \mathfrak{w}_i\, \eta_i$$

$$= \begin{pmatrix} \langle\sqrt[q]{1-\prod_{i=1}^m(1-\ {}^i\Psi_{\breve{\mathfrak{X}}}^q)^{\mathfrak{w}_i}}, \prod_{i=1}^m(\ {}^i\Theta_{\breve{\mathfrak{X}}})^{\mathfrak{w}_i}\rangle, \\ \langle\sqrt[q]{1-\prod_{i=1}^m(1-\ {}^i\alpha_{\breve{\mathfrak{X}}}^q)^{\mathfrak{w}_i}}, \prod_{i=1}^m(\ {}^i\beta_{\breve{\mathfrak{X}}})^{\mathfrak{w}_i}\rangle \end{pmatrix} \tag{2.4}$$

where $q \geq 1$.

3. EXPONENTIAL SIMILARITY MEASURES FOR q-RUNG LINEAR DIOPHANTINE FUZZY SETS

In this part, we describe the similarity measures between two q-RLDFSs based on the exponential function (EF) and discuss their some properties.

Since an EF $e^{-\lambda}$ is decreasing when λ lies within $[0,\infty)$, its function value is $(0,1]$. If we consider λ as a distance measure between two objects based on the mathematical property of the EF $e^{-\lambda}$ for $\lambda \in [0,\infty)$, we can describe the exponential similarity measures between the q-RLDFSs.

Definition 3.1. Let $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ be two q-RLDFSs over $\mathfrak{D}$. Then, the exponential similarity measure (ESM) between two q-RLDFSs $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ is defined as

$$S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = \frac{1}{n}\sum_{k=1}^n e^{-\begin{pmatrix} \left|\Psi_{\breve{\mathfrak{X}}_1}(d_k) - \Psi_{\breve{\mathfrak{X}}_2}(d_k)\right| + \left|\Theta_{\breve{\mathfrak{X}}_1}(d_k) - \Theta_{\breve{\mathfrak{X}}_2}(d_k)\right| \\ + \left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k) - \alpha^q_{\breve{\mathfrak{X}}_2}(d_k)\right| + \left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k) - \beta^q_{\breve{\mathfrak{X}}_2}(d_k)\right| \end{pmatrix}}. \tag{3.1}$$

Proposition 3.2. Let $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ be two q-RLDFSs over $\mathfrak{D}$. Then, the proposed ESM $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2)$ satisfies the following features:

i. $0 < S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) \leq 1$.
ii. $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = 1$ if and only if $\breve{\mathfrak{X}}_1 = \breve{\mathfrak{X}}_2$.
iii. $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = S_e(\breve{\mathfrak{X}}_2, \breve{\mathfrak{X}}_1)$.

iv. If $\breve{\mathfrak{X}}_1 \subseteq \breve{\mathfrak{X}}_2 \subseteq \breve{\mathfrak{X}}_3$ for the q-RLDFS $\breve{\mathfrak{X}}_3$ then $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_3) \leq S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2)$ and $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_3) \leq S_e(\breve{\mathfrak{X}}_2, \breve{\mathfrak{X}}_3)$.

Proof.
i. It is clear that this property is valid.

ii. ⇒: Let $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = 1$. Considering Equation (3.1), we can say that $e^{-\left(\left|\Psi_{\breve{\mathfrak{X}}_1}(d_k)-\Psi_{\breve{\mathfrak{X}}_2}(d_k)\right|+\left|\Theta_{\breve{\mathfrak{X}}_1}(d_k)-\Theta_{\breve{\mathfrak{X}}_2}(d_k)\right|+\left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k)-\alpha^q_{\breve{\mathfrak{X}}_1}(d_k)\right|+\left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k)-\beta^q_{\breve{\mathfrak{X}}_1}(d_k)\right|\right)} = 1$. This implies that $\Psi_{\breve{\mathfrak{X}}_1}(d_k) = \Psi_{\breve{\mathfrak{X}}_2}(d_k)$, $\Theta_{\breve{\mathfrak{X}}_1}(d_k) = \Theta_{\breve{\mathfrak{X}}_2}(d_k)$, $\alpha_{\breve{\mathfrak{X}}_1}(d_k) = \alpha_{\breve{\mathfrak{X}}_2}(d_k)$, $\beta_{\breve{\mathfrak{X}}_1}(d_k) = \beta_{\breve{\mathfrak{X}}_2}(d_k)$ for each $d_k \in \mathfrak{D}$ since $e^0 = 1$. Obviously, we deduce that $\breve{\mathfrak{X}}_1 = \breve{\mathfrak{X}}_2$.
⇐: Let $\breve{\mathfrak{X}}_1 = \breve{\mathfrak{X}}_2$. By Definition 2.3, we have $\Psi_{\breve{\mathfrak{X}}_1}(d_k) - \Psi_{\breve{\mathfrak{X}}_2}(d_k) = 0$, $\Theta_{\breve{\mathfrak{X}}_1}(d_k) - \Theta_{\breve{\mathfrak{X}}_2}(d_k) = 0$, $\alpha_{\breve{\mathfrak{X}}_1}(d_k) - \alpha_{\breve{\mathfrak{X}}_2}(d_k) = 0$ and $\beta_{\breve{\mathfrak{X}}_1}(d_k) - \beta_{\breve{\mathfrak{X}}_2}(d_k) = 0$ for all $d_k \in \mathfrak{D}$. (It is clear that $\alpha^q_{\breve{\mathfrak{X}}_1}(d_k) - \alpha^q_{\breve{\mathfrak{X}}_2}(d_k) = 0$ and $\beta^q_{\breve{\mathfrak{X}}_1}(d_k) - \beta^q_{\breve{\mathfrak{X}}_2}(d_k) = 0$ for $q \geq 1$). Hence, we conclude that $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = 1$.

iii. This property is clear from Equation (3.1).

iv. If $\breve{\mathfrak{X}}_1 \subseteq \breve{\mathfrak{X}}_2 \subseteq \breve{\mathfrak{X}}_3$ then by using Definition 2.3, we calculate $\left|\Psi_{\breve{\mathfrak{X}}_1}(d_k) - \Psi_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\Psi_{\breve{\mathfrak{X}}_1}(d_k) - \Psi_{\breve{\mathfrak{X}}_2}(d_k)\right|$, $\left|\Theta_{\breve{\mathfrak{X}}_1}(d_k) - \Theta_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\Theta_{\breve{\mathfrak{X}}_1}(d_k) - \Theta_{\breve{\mathfrak{X}}_2}(d_k)\right|$, $\left|\alpha_{\breve{\mathfrak{X}}_1}(d_k) - \alpha_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\alpha_{\breve{\mathfrak{X}}_1}(d_k) - \alpha_{\breve{\mathfrak{X}}_2}(d_k)\right|$ and $\left|\alpha_{\breve{\mathfrak{X}}_1}(d_k) - \alpha_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\beta_{\breve{\mathfrak{X}}_1}(d_k) - \beta_{\breve{\mathfrak{X}}_2}(d_k)\right|$ (i.e., $\left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k) - \alpha^q_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k) - \alpha^q_{\breve{\mathfrak{X}}_2}(d_k)\right|$ and $\left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k) - \beta^q_{\breve{\mathfrak{X}}_3}(d_k)\right| \geq \left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k) - \beta^q_{\breve{\mathfrak{X}}_2}(d_k)\right|$) for all $d_k \in \mathfrak{D}$. Thus, we have $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_3) \leq S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2)$ since the EF $e^{-\lambda}$ is decreasing for $\lambda \in [0, \infty)$. By using a similar technique, we can prove that $S_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_3) \leq S_e(\breve{\mathfrak{X}}_2, \breve{\mathfrak{X}}_3)$. The proof is completed.

Besides this, we can also describe generalized types of the ESM based on Equation (3.1) as follows.

Definition 3.2. Let $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ be two q-RLDFSs over $\mathfrak{D}$. Then, the generalized exponential similarity measure (GESM) among the q-RLDFSs $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ is defined as

$$S^p_e(\breve{\mathfrak{X}}_1, \breve{\mathfrak{X}}_2) = \frac{1}{n}\sum_{k=1}^{n} e^{-\left(\begin{array}{c}\left|\Psi_{\breve{\mathfrak{X}}_1}(d_k)-\Psi_{\breve{\mathfrak{X}}_2}(d_k)\right|^p+\left|\Theta_{\breve{\mathfrak{X}}_1}(d_k)-\Theta_{\breve{\mathfrak{X}}_2}(d_k)\right|^p \\ +\left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k)-\alpha^q_{\breve{\mathfrak{X}}_2}(d_k)\right|^p+\left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k)-\beta^q_{\breve{\mathfrak{X}}_2}(d_k)\right|^p\end{array}\right)} \tag{3.2}$$

for all $p \in \mathbb{N} = \{1,2,\ldots\}$.

We observe that if we put $p = 1$ in Definition 3.2 then the GESM reduces to the ESM (in Definition 3.1). That is, the ESM is a special form of the GESM.

It is expected that Equation (3.2) for variations of values of p can present many advantages in practice.

Obviously, the GESM satisfies the properties (i-iv) in Proposition 3.2.

In practical applications, one can consider different weights for the elements in the q-RLDFS. Suppose that the weight vector of the elements is $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n)^T$ where $\mathfrak{w}_k \in (0,1]$ and $\sum_{k=1}^{n} \mathfrak{w}_k = 1$. Now, we can describe the weighted GESM for q-RLDFSs as follows.

Definition 3.3. Let $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ be two q-RLDFSs over $\mathfrak{D}$. Then, the generalized exponential similarity measure (GESM) among the q-RLDFSs $\breve{\mathfrak{X}}_1$ and $\breve{\mathfrak{X}}_2$ is defined as

$$S^p_{e_{\mathfrak{w}}}(\breve{\mathfrak{X}}_1,\breve{\mathfrak{X}}_2)=\sum_{k=1}^{n}\mathfrak{w}_k\, e^{-\left(\begin{array}{c}\left|\Psi_{\breve{\mathfrak{X}}_1}(d_k)-\Psi_{\breve{\mathfrak{X}}_2}(d_k)\right|^p+\left|\Theta_{\breve{\mathfrak{X}}_1}(d_k)-\Theta_{\breve{\mathfrak{X}}_2}(d_k)\right|^p\\+\left|\alpha^q_{\breve{\mathfrak{X}}_1}(d_k)-\alpha^q_{\breve{\mathfrak{X}}_2}(d_k)\right|^p+\left|\beta^q_{\breve{\mathfrak{X}}_1}(d_k)-\beta^q_{\breve{\mathfrak{X}}_2}(d_k)\right|^p\end{array}\right)} \tag{3.3}$$

for all $p\in\mathbb{N}=\{1,2,\dots\}$.

Obviously, the weighted GESM satisfies the properties (i-iv) in Proposition 3.2.

4. A CLUSTERING ALGORITHM BASED ON SIMILARITY MEASURES OF q-RUNG LINEAR DIOPHANTINE FUZZY SETS WITH APPLICATION

In this part, we construct a clustering algorithm (CA) to illustrate the performance of SMs of q-RLDFSs and apply it to the clustering problem.

The complete step-by-step procedure of q-rung linear Diophantine fuzzy clustering (q-RLDFC) algorithm is given as follows.

Algorithm (q-RLDFC Algorithm)

Input: Given a set of q-RLDFSs $\{\breve{\mathfrak{X}}_1,\breve{\mathfrak{X}}_2,\dots,\breve{\mathfrak{X}}_t\}$

Output: Clusters of the q-RLDFSs

Step 1: Calculate $S^p_e(\breve{\mathfrak{X}}_r,\breve{\mathfrak{X}}_s)$ for each r,s $(r\neq s)$
(If there are weights, then calculate $S^p_{e_{\mathfrak{w}}}(\breve{\mathfrak{X}}_r,\breve{\mathfrak{X}}_s)$)

Step 2: Fuse two clusters with greater similarity coefficient
(The procedure is repeated time and again until the desirable number of clusters is achieved. Only two clusters can be fused in each stage and they cannot be separated after they are fused)

Step 3: Compute the average ($\mathfrak{A}$) of each cluster obtained in Step 2 by using Equation (2.4)

Step 4: Compare each cluster with the other clusters by using the (weighted) GESM

Step 5: Cluster the q-RLDFSs into the different clusters

The q-RLDFC Algorithm is used to deal with clustering problems as discussed in the example below.

Example 4.1. We consider the experimental data for five cars $\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3$, $\mathfrak{C}_4$ and $\mathfrak{C}_5$. The performance of the each car is determined by the four features with low cost f_1: design, f_2: comfort, f_3: safety, and f_4: aerodynamic. It is obvious that there is a cost to have each feature in the car. Therefore, we consider these four features with low cost. For the construction of q-RLDFSs, we take the reference parameter as "price" (e.g., cheap and not cheap/expensive). That is, we consider the problem of obtaining the order of selection of cars under features associated with the reference parameter (price). Thus, the q-RLDFSs for each car are given as follows:

$$\mathfrak{C}_1=\left\{\begin{array}{l}(f_1,\langle 0.7,0.4\rangle,\langle 0.5,0.7\rangle),(f_2,\langle 0.3,0.6\rangle,\langle 0.2,0.9\rangle),\\(f_3,\langle 0.9,0.6\rangle,\langle 0.3,0.5\rangle),(f_4,\langle 0.8,0.7\rangle,\langle 0.7,0.8\rangle)\end{array}\right\},$$

$$\mathfrak{C}_2=\left\{\begin{array}{l}(f_1,\langle 0.5,0.8\rangle,\langle 0.4,0.4\rangle),(f_2,\langle 0.7,0.1\rangle,\langle 0.6,0.3\rangle),\\(f_3,\langle 0.2,0.9\rangle,\langle 0.3,0.7\rangle),(f_4,\langle 0.3,0.3\rangle,\langle 0.5,0.2\rangle)\end{array}\right\},$$

$$\mathfrak{C}_3=\left\{\begin{array}{l}(f_1,\langle 0.2,0.8\rangle,\langle 0.4,0.6\rangle),(f_2,\langle 0.9,0.7\rangle,\langle 0.7,0.7\rangle),\\(f_3,\langle 0.5,0.1\rangle,\langle 0.5,0.1\rangle),(f_4,\langle 0.4,0.5\rangle,\langle 0.6,0.3\rangle)\end{array}\right\},$$

$$\mathfrak{C}_4=\left\{\begin{matrix}(f_1,\langle 0.1,0.7\rangle,\langle 0.3,0.1\rangle),(f_2,\langle 0.6,0.4\rangle,\langle 0.6,0.5\rangle),\\(f_3,\langle 0.4,0.5\rangle,\langle 0.3,0.8\rangle),(f_4,\langle 0.9,0.6\rangle,\langle 0.9,0.5\rangle)\end{matrix}\right\},$$

$$\mathfrak{C}_5=\left\{\begin{matrix}(f_1,\langle 0.6,0.6\rangle,\langle 0.3,0.4\rangle),(f_2,\langle 0.7,0.2\rangle,\langle 0.5,0.5\rangle),\\(f_3,\langle 0.8,0.4\rangle,\langle 0.6,0.2\rangle),(f_4,\langle 0.9,0.0\rangle,\langle 0.7,0.4\rangle)\end{matrix}\right\}.$$

Since $0.7^3+0.8^3\leq 1$ we can take $q=3$. Also, we assume that the weight vector $\mathfrak{w}=(0.2,0.5,0.1,0.2)^T$. Now, the steps of q-RLDFC Algorithm are discussed as below.

In the first stage, each of the q-RLDFSs $\mathfrak{C}_r$ $(r=1,2,3,4,5)$ is considered as a unique cluster $\{\mathfrak{C}_1\},\{\mathfrak{C}_2\},\{\mathfrak{C}_3\},\{\mathfrak{C}_4\},\{\mathfrak{C}_5\}$.

Step 1: We calculate the weighted GESMs for $p=2$ (arbitrarily chosen) as in Table 1.

Table 1. *The weighted GESMs $S^2_{e_{\mathfrak{w}}}$ for $p=2$*

$S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_r,\mathfrak{C}_s)$	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
$\mathfrak{C}_1$	null	0.496622	0.588173	0.616088	0.612765
$\mathfrak{C}_2$	0.496622	null	0.708138	0.782837	0.844325
$\mathfrak{C}_3$	0.588173	0.708138	null	0.76056	0.702033
$\mathfrak{C}_4$	0.616088	0.782837	0.76056	null	0.80821
$\mathfrak{C}_5$	0.612765	0.844325	0.702033	0.80821	null

Step 2: From Table 1, $S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_2,\mathfrak{C}_5)=0.844325$ is the greater similarity coefficient than the others. Hence, we fuse the clusters $\{\mathfrak{C}_2\}$ and $\{\mathfrak{C}_5\}$. Then, the q-RLDFSs $\mathfrak{C}_r$ $(r=1,2,3,4,5)$ clusters into the four clusters at this stage

$$\{\mathfrak{C}_1\},\{\mathfrak{C}_2,\mathfrak{C}_5\},\{\mathfrak{C}_3\},\{\mathfrak{C}_4\}.$$

Note that $S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_5,\mathfrak{C}_4)=0.80821>S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_3,\mathfrak{C}_4)=0.76056>S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_4,\mathfrak{C}_5)=0.616088$. Then, $\{\mathfrak{C}_5\}$ is the best option for fusing $\{\mathfrak{C}_4\}$. But $\{\mathfrak{C}_2\}$ is the best option for fusing $\{\mathfrak{C}_5\}$ since $S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_2,\mathfrak{C}_5)=0.844325>S^2_{e_{\mathfrak{w}}}(\mathfrak{C}_4,\mathfrak{C}_5)=0.80821$. At this stage, fusing $\{\mathfrak{C}_4\}$ with $\{\mathfrak{C}_3\}$ is not reasonable considering the similarity coefficients.

Step 3: The average of each cluster employing Equation (2.4) are calculated as follows:
$\mathfrak{A}(\mathfrak{C}_1)=\mathfrak{C}_1$, $\mathfrak{A}(\mathfrak{C}_3)=\mathfrak{C}_3$, $\mathfrak{A}(\mathfrak{C}_4)=\mathfrak{C}_4$ and

$$\mathfrak{A}(\mathfrak{C}_2,\mathfrak{C}_5)=q-RLDFWAA_{\mathfrak{w}}(\mathfrak{C}_2,\mathfrak{C}_5)=\left\{\begin{matrix}(f_1,\langle 0.42,0.86\rangle,\langle 0.26,0.69\rangle),(f_2,\langle 0.7,0.14\rangle,\langle 0.56,0.39\rangle),\\(f_3,\langle 0.41,0.9\rangle,\langle 0.3,0.82\rangle),(f_4,\langle 0.62,0\rangle,\langle 0.47,0.6\rangle)\end{matrix}\right\}.$$

Step 4: We equate each cluster with the other clusters by using the weighted GESM $S^2_{e_{\mathfrak{w}}}$ as follows.

$$S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1),\mathfrak{A}(\mathfrak{C}_2,\mathfrak{C}_5)\big)=0.5245666,\ S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1),\mathfrak{A}(\mathfrak{C}_3)\big)=0.5881731,$$
$$S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1),\mathfrak{A}(\mathfrak{C}_4)\big)=0.6160884,\ S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2,\mathfrak{C}_5),\mathfrak{A}(\mathfrak{C}_3)\big)=0.6817248,$$
$$S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2,\mathfrak{C}_5),\mathfrak{A}(\mathfrak{C}_4)\big)=0.7896134,\ S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_3),\mathfrak{A}(\mathfrak{C}_4)\big)=0.7605597.$$

Since $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2,\mathfrak{C}_5),\mathfrak{A}(\mathfrak{C}_4)\big)=0.7896134$ is the greater similarity coefficient than the others, we fuse the clusters $\{\mathfrak{C}_2,\mathfrak{C}_5\}$ and $\{\mathfrak{C}_4\}$. Then, the q-RLDFSs $\mathfrak{C}_r(r=1,2,3,4,5)$ are clustered into the three clusters at this stage

$$\{\mathfrak{C}_1\}, \{\mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5\}, \{\mathfrak{C}_3\}.$$

Also, we note that $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_3), \mathfrak{A}(\mathfrak{C}_4)\big) > S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2, \mathfrak{C}_5), \mathfrak{A}(\mathfrak{C}_3)\big) > S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1), \mathfrak{A}(\mathfrak{C}_3)\big)$.

Repeating Step 3 and Step 4, we obtain $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1), \mathfrak{A}(\mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5)\big) = 0.4958683$, $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_1), \mathfrak{A}(\mathfrak{C}_3)\big) = 0.5881731$ and $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5), \mathfrak{A}(\mathfrak{C}_3)\big) = 0.6678252$.

Since $S^2_{e_{\mathfrak{w}}}\big(\mathfrak{A}(\mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5), \mathfrak{A}(\mathfrak{C}_3)\big) = 0.6678252$ is the greater similarity coefficient than the others, we fuse the clusters $\{\mathfrak{C}_2, \mathfrak{C}_4, \mathfrak{C}_5\}$ and $\{\mathfrak{C}_3\}$. Then, the q-RLDFSs $\mathfrak{C}_r$ $(r = 1,2,3,4,5)$ are clustered into the two clusters at this stage

$$\{\mathfrak{C}_1\}, \{\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}.$$

Lastly, the two clusters presented above are clustered into a unique cluster as

$$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}.$$

The above clustering processes of cars by our proposed q-RLDFC Algorithm using $S^2_{e_{\mathfrak{w}}}$ can be illustrated as in Figure 2.

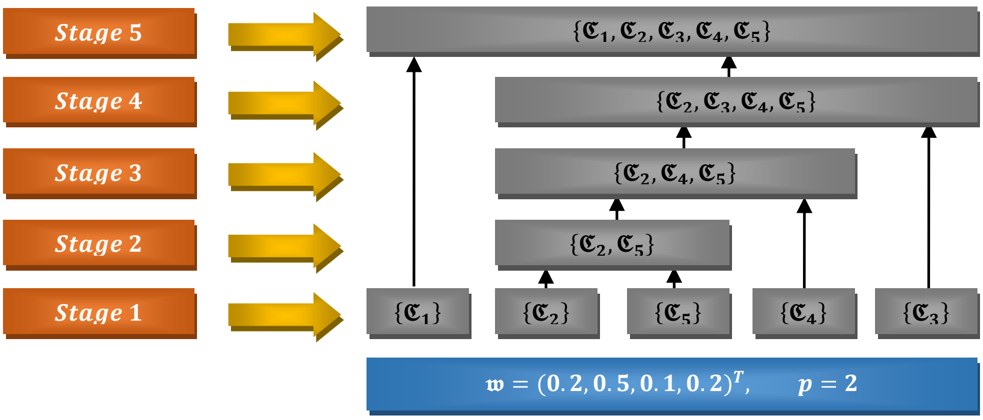

Figure 2. *The clustering tree of cars through the proposed q-RLDFC Algorithm*

The clustering processes of cars given in Example 4.1 for different values of $\mathfrak{w}$ and p by using the proposed q-RLDFC Algorithm can be illustrated as in Figure 3.

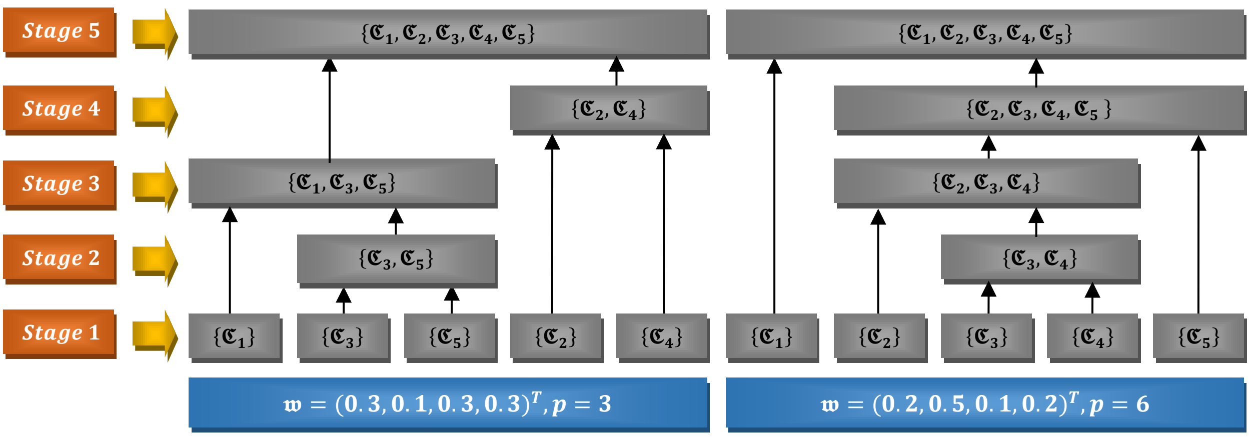

Figure 3. *The clustering tree of cars through different* $\mathfrak{w}$ *and* p *for Example 4.1*

5. SUPERIORITY AND COMPARATIVE ANALYSIS

The q-RLDFSs cover the space of IFS, PFS, q-ROFS and LDFS. The qth power to parameters enlarges the ambit of membership and non-membership degrees. To demonstrate the superiority as well as the advantages of emerging SMs, we match their performance with those of some existing SMs S_{HY1}, S_Z based on IFS, PyFS respectively and q-ROFC[1], q-ROFC[2] based on q-ROFS on some common data sets. Consider the intuitionistic fuzzy elements M and N from Case 1 to Case 4 in Table 2. From Table 2, we understand that the SM S_{HY1} [17] in IFSs is utilized between M and N. The results are inconsistent since Case 3 and Case 4 are same. Since the space of PyFSs is larger than IFSs. It enables us to find the SM between M and N by using the SM S_Z [19] in PyFSs. When Case 1 and Case 2 are considered, the result is inconsistent. Again, since the space of q-ROFSs is wider than IFSs, the SM between M and N is carried out by the SMs $q - ROFC^i$ $(q = 2)$ $(i = 1, 2)$ [24] in q-ROFSs. The result is inconsistent as NaN (Not a Number) occurs for Case 1 and Case 2. It is clear that (α, β), the number form of IFSs, PyFSs and q-ROFSs can be viewed as $(0, 0, \alpha, \beta)$, the number form of q-RLDFSs. It leads to find the SM between M and N by applying our proposed GESM S_e^p for $p = 2$. We get a convenient result as shown in Table 2. It proves that our proposed SM is superior than the existing SMs.

Table 2. *Comparison of the GESM S_e^p for $p = 2$ (Data in Table 3 in [25])*

	Case 1	**Case 2**	**Case 3**	**Case 4**
M	$\{(x, 0.5, 0.5)\}$	$\{(x, 0.6, 0.4)\}$	$\{(x, 0, 0.87)\}$	$\{(x, 0.6, 0.27)\}$
N	$\{(x, 0, 0)\}$	$\{(x, 0, 0)\}$	$\{(x, 0.28, 0.55)\}$	$\{(x, 0.28, 0.55)\}$
S_{HY1} [17]	0.3333	0.25	**0.5152**	**0.5152**
S_Z [19]	**0.5**	**0.5**	0.5989	0.1696
q-ROFC[1] $(q = 2)$ [24]	**NaN**	**NaN**	0.999961	0.012599
q-ROFC[2] $(q = 2)$ [24]	**NaN**	**NaN**	0.999971	0.509326
The GESM S_e^2 $(q = 2)$	0.6065307	0.5945205	0.5869592	0.5997754

In this table, NaN means Not a Number.

6. RESULTS

In this paper, we proposed the SMs based on the EF to determine similarity among two q-RLDFSs. By using these developed SMs, we constructed a CA and addressed a real-life clustering problem in q-RLDF setting. In addition, we presented comparison results showing that the proposed SMs yield more convincing results than the existing SMs and can be applied to issues such as medical diagnosis and pattern recognition. In conclusion, the main contributions are reviewed and illustrated in the following.

(1) The formulas of ESMs of q-RLDFs are established, and their desirable properties are studied. Meanwhile, several relations between the proposed SMs have also been elicited.

(2) A comparison with several SMs in literature is offered in Table 2 to indicate the availability of the new SMs.

We hope that the findings in this article will offer new perspectives to researchers addressing various real world issues. Our future venture is to explore new similarity, correlation, and distance measures based on q-RLDFSs. Thus, we can further expand the application range of q-RLDFSs.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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